

MAXIMAL TORUS THEORY FOR COMPACT QUANTUM GROUPS

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ABSTRACT. Associated to any compact quantum group $G \subset U_N^+$ is a canonical family of group dual subgroups $\widehat{\Gamma}_Q \subset G$, parametrized by unitaries $Q \in U_N$, playing the role of “maximal tori” for G . We present here a series of conjectures, relating the various algebraic and analytic properties of G to those of the family $\{\widehat{\Gamma}_Q | Q \in U_N\}$.

INTRODUCTION

We investigate here the notion of “maximal torus” for the compact quantum groups. In general, the maximal torus does not really exist. Given a closed subgroup $G \subset U_N^+$, what does exist, however, is a family of group dual subgroups $\widehat{\Gamma}_Q \subset G$, parametrized by unitaries $Q \in U_N$, which altogether play the role of the maximal torus.

The construction, which goes back to [1], [6], is very simple, as follows:

$$C^*(\Gamma_Q) = C(G) / \left\langle (QuQ^*)_{ij} = 0, \forall i \neq j \right\rangle$$

Here u is the fundamental corepresentation of G , and the key observation is that the elements $g_i = (QuQ^*)_{ii}$ are group-like in the quotient algebra on the right.

Based on growing evidence, coming from the recent quantum group literature, we will formulate here a series of conjectures, relating the various algebraic and analytic properties of G to those of the family $\{\widehat{\Gamma}_Q | Q \in U_N\}$. Our main statements are as follows:

- (1) Character conjecture: Assuming that G is connected, any nonzero element $P \in C(G)_{central}$ is left nonzero in one of the quotients $C^*(\Gamma_Q)$.
- (2) Amenability conjecture: The discrete quantum group \widehat{G} is amenable if and only if all the discrete groups Γ_Q are amenable.
- (3) Growth conjecture: The discrete quantum group \widehat{G} has polynomial growth if and only if each Γ_Q has polynomial growth.

We believe all these conjectures to be true, non-trivial, and of course, of interest.

The paper is organized as follows: 1-2 are preliminary sections, in 3-4 we comment on the above conjectures, and in 5-6 we discuss Tannakian aspects.

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1. MAXIMAL TORI

We use Woronowicz's compact quantum group formalism in [29], [30], with the extra axiom $S^2 = id$. The precise definition that we will need is as follows:

Definition 1.1. *Assume that A is a C^* -algebra, and $u \in M_N(A)$ is a unitary matrix, such that the following formulae define morphisms of C^* -algebras:*

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj} \quad , \quad \varepsilon(u_{ij}) = \delta_{ij} \quad , \quad S(u_{ij}) = u_{ji}^*$$

We write then $A = C(G)$, and call G a compact matrix quantum group.

The above maps Δ, ε, S are called comultiplication, counit and antipode. The basic examples include the compact Lie groups $G \subset U_N$, their q -deformations at $q = -1$, and the duals of the finitely generated discrete groups $\Gamma = \langle g_1, \dots, g_N \rangle$. See [20], [29].

We have as well the following key construction, due to Wang [27]:

Definition 1.2. *The quantum groups O_N^+, U_N^+ constructed via*

$$\begin{aligned} C(O_N^+) &= C^* \left((u_{ij})_{i,j=1,\dots,N} \middle| u = \bar{u}, u^t = u^{-1} \right) \\ C(U_N^+) &= C^* \left((u_{ij})_{i,j=1,\dots,N} \middle| u^* = u^{-1}, u^t = \bar{u}^{-1} \right) \end{aligned}$$

are called free analogues of O_N, U_N .

Here the existence of the morphisms Δ, ε, S , as in Definition 1.1, comes from the universality property of the above algebras. Observe that by dividing by the commutator ideal, we obtain respectively the algebras $C(O_N), C(U_N)$. Thus, we have inclusions $O_N \subset O_N^+, U_N \subset U_N^+$. These inclusions are far from being isomorphisms. See [27].

The notion of diagonal subgroup goes back to [6]. The idea is very simple:

Definition 1.3. *Given a closed subgroup $G \subset U_N^+$, we set*

$$C^*(\Gamma_1) = C(G) / \left\langle u_{ij} = 0, \forall i \neq j \right\rangle$$

with $u \in M_N(C(G))$ being the fundamental corepresentation.

As explained in [6], the above quotient algebra is indeed cocommutative, and its generators $g_i = u_{ii}$ are group-like. Thus, $\Gamma_1 = \langle g_1, \dots, g_N \rangle$ is a usual discrete group. Observe that $C^*(\Gamma_1)$ is not necessarily the full group algebra of Γ_1 , but rather a certain quotient of it. For simplicity of presentation, we will use however the notation $C^*(\Gamma_1)$.

Observe that in the classical case, $G \subset U_N$, we obtain in this way the dual of the diagonal torus, $\Gamma_1 = \widehat{G} \cap \mathbb{T}^N$, with $\mathbb{T}^N \subset U_N$ being the group of diagonal unitaries.

A key extension of the above construction, obtained by using a “spinning” matrix $Q \in U_N$, was proposed in [1]. The idea is once again very simple, as follows:

Definition 1.4. Given a closed subgroup $G \subset U_N^+$, and a matrix $Q \in U_N$, we set

$$C^*(\Gamma_Q) = C(G) / \left\langle v_{ij} = 0, \forall i \neq j \right\rangle$$

where $v = QuQ^*$, with $u \in M_N(C(G))$ being the fundamental corepresentation.

Observe that for the identity matrix $Q = 1$ we obtain indeed the discrete group Γ_1 from Definition 1.3. In general, we do not really have a new notion here, because Γ_Q is nothing but the group Γ_1 from Definition 1.3, taken for the quantum group (G, v) .

The theoretical interest in this slight generalization of Definition 1.3 comes from the following fundamental result, due to Woronowicz [29]:

Theorem 1.5. Any group dual subgroup $\hat{\Lambda} \subset G$ must appear as

$$\hat{\Lambda} \subset \hat{\Gamma}_Q \subset G$$

for a certain matrix $Q \in U_N$.

Proof. As explained in [29], the finite dimensional unitary corepresentations of $\hat{\Lambda}$ are completely reducible, with the irreducible corepresentations being all 1-dimensional, corresponding to the group elements $g \in \Gamma$. Thus, such a corepresentation must be of the form $v = QwQ^*$, with $w = \text{diag}(g_1, \dots, g_N)$ and $Q \in U_N$. We conclude that the embeddings $\hat{\Lambda} \subset U_N^+$ come from the quotient maps $C(U_N^+) \rightarrow C^*(\Lambda)$ of type $u \rightarrow QwQ^*$, and so the subgroups $\hat{\Lambda} \subset G \subset U_N^+$ must appear as in the statement. See [1], [29]. \square

We have as well the following related result, from [6]:

Proposition 1.6. If the elements $g_1, \dots, g_N \in \Gamma_Q$ are pairwise distinct, then $\hat{\Gamma}_Q \subset G$ is a maximal group dual subgroup.

Proof. By rotating, we can assume $Q = 1$. Given a subgroup $\hat{\Gamma}_1 \subset \hat{\Lambda} \subset G$, let us denote by $w \leftarrow v \leftarrow u$ the corresponding fundamental corepresentations. We have then $v = P \text{diag}(h_1, \dots, h_N) P^*$, for a certain $P \in U_N$, where h_1, \dots, h_N are the generators of Λ . In addition, the quotient map $\Lambda \rightarrow \Gamma_1$ must send $h_i \rightarrow g_{\sigma(i)}$, for a certain permutation $\sigma \in S_N$. We deduce that $w = \text{diag}(g_1, \dots, g_N)$ commutes with $R = \sigma P$, which reads $R_{ij}(g_i - g_j) = 0$, and so $R_{ij} = 0$ for $i \neq j$. But this gives $\Lambda = \Gamma_1$. See [6]. \square

The considerations in [6] were motivated by the root systems for half-classical quantum groups. The general problem here, still open, can benefit from the systematic approach to half-liberation in [8]. We will be back to these topics in section 5 below.

As for the considerations in [1], these were motivated by a key rigidity conjecture in noncommutative geometry [16], solved in the meantime by Goswami and Joardar [17]. The main observation in [1] was the fact that a non-classical group dual cannot act on a compact Riemannian manifold. In view of [17], this is of course obsolete.

2. BASIC EXAMPLES

In this section we discuss, following [1], some basic examples of the construction $(G, Q) \rightarrow \Gamma_Q$ from Definition 1.4. In the classical case, the result is as follows:

Proposition 2.1. *For a closed subgroup $G \subset U_N$ we have*

$$\widehat{\Gamma}_Q = G \cap (Q^* \mathbb{T}^N Q)$$

where $\mathbb{T}^N \subset U_N$ is the group of diagonal unitary matrices.

Proof. This is indeed clear at $Q = 1$, where Γ_1 appears by definition as the dual of the compact abelian group $G \cap \mathbb{T}^N$. In general, this follows by conjugating by Q . \square

Here are as well the computations for Wang's quantum groups in [27]:

Proposition 2.2. *The construction $G \rightarrow \Gamma_Q$ is as follows:*

- (1) *For $G = U_N^+$ we obtain $\Gamma_Q = F_N$, for any $Q \in U_N$.*
- (2) *For $G = O_N^+$ we have $\Gamma_Q = F_N / \langle R_{ij} \neq 0 \implies g_i g_j = 1 \rangle$, where $R = QQ^t$.*

Proof. These results are well-known, the proof being as follows:

(1) At $Q = 1$ this is clear, and in the general case $Q \in U_N$ this follows from $\Gamma_1 = F_N$, and from the fact that $C(U_N^+)$ is isomorphic to itself via $u \rightarrow QuQ^*$.

(2) At $Q = 1$ this is clear, and we obtain the group $\Gamma_1 = \mathbb{Z}_2^{*N}$. In general now, with $v = QuQ^*$, and with $R = QQ^t$ as in the statement, we have:

$$\bar{v} = \overline{QuQ^*} = \bar{Q}\bar{u}Q^t = \bar{Q}uQ^t = \bar{Q}Q^*vQQ^t = R^*vR$$

Thus Γ_Q is presented by the relations $\bar{w} = R^*wR$, with $w = \text{diag}(g_1, \dots, g_N)$. But $(wR)_{ij} = (R\bar{w})_{ij}$ with reads $g_i R_{ij} = R_{ij} g_j^{-1}$, and this gives the result. \square

Following [1], we have as well the following result:

Proposition 2.3. *When $\Gamma = \widehat{G}$ is classical, diagonally embedded, we have:*

$$\Gamma_Q = \Gamma / \langle g_i = g_j | \exists k, Q_{ki} \neq 0, Q_{kj} \neq 0 \rangle$$

A similar result holds for the arbitrary (non-diagonal) embeddings.

Proof. Assume indeed that $\Gamma = \langle g_1, \dots, g_N \rangle$ is a discrete group, with $\widehat{\Gamma} \subset U_N^+$ coming via $u = \text{diag}(g_1, \dots, g_N)$. With $v = QuQ^*$, we have:

$$\sum_s \bar{Q}_{si} v_{sk} = \sum_{st} \bar{Q}_{si} Q_{st} \bar{Q}_{kt} g_t = \sum_t \delta_{it} \bar{Q}_{kt} g_t = \bar{Q}_{ki} g_i$$

Thus the condition $v_{ij} = 0$ for $i \neq j$ gives $\bar{Q}_{ki} v_{kk} = \bar{Q}_{ki} g_i$, which is the same as saying that $Q_{ki} \neq 0$ implies $g_i = v_{kk}$. But this latter equality reads $g_i = \sum_j |Q_{kj}|^2 g_j$, and we conclude that $Q_{ki} \neq 0, Q_{kj} \neq 0$ implies $g_i = g_j$, as desired. See [1]. \square

In what follows we will regard the family $\{\Gamma_Q | Q \in U_N\}$ as a kind of “bundle” over the group U_N . Observe that, by Proposition 2.3, we cannot expect the correspondence $Q \rightarrow \Gamma_Q$ to have any reasonable continuity property, and so our “bundle” structure to fit into some known formalism. In addition, we have the following negative result:

Proposition 2.4. *When $\Gamma = \widehat{G}$ is a classical group, the fibers Γ_Q are trivial (in the sense that they are quotients of \mathbb{Z}), for generic values of $Q \in U_N$.*

Proof. This follows indeed from Proposition 2.3. To be more precise, we obtain that Γ_Q is trivial, with probability 1, with respect to the Haar measure on U_N . \square

The above result cuts short any attempt of using probabilistic tools, in order to “average” our family of groups $\{\Gamma_Q | Q \in U_N\}$. Would the fibers have been generically non-trivial, we could have probably used [12] in order to average the various numeric invariants of Γ_Q , in order to obtain a unique, formal “maximal torus”. But, this is not the case.

Following now [14], we have as well the following result, which can provide counterexamples to some other various naive conjectures which can be made:

Proposition 2.5. *For the quantum group of Kac-Paljutkin, and for its generalizations by Sekine, the family $\{\Gamma_Q | Q \in U_N\}$ consists of abelian groups.*

Proof. This follows from the results of Franz and Skalski in [14], who classified all the group dual subgroups of these quantum groups, from [25]. See [1]. \square

Finally, we have the following result, regarding Wang’s free analogue of the symmetric group [28], which combines various findings from [1], [7], and which is perhaps the most illustrating, for the various phenomena that can appear:

Theorem 2.6. *For the quantum permutation group $G = S_N^+$, we have:*

- (1) *Given $Q \in U_N$, the quotient $F_N \rightarrow \Gamma_Q$ comes from the following relations:*

$$\begin{cases} g_i = 1 & \text{if } \sum_l Q_{il} \neq 0 \\ g_i g_j = 1 & \text{if } \sum_l Q_{il} Q_{jl} \neq 0 \\ g_i g_j g_k = 1 & \text{if } \sum_l Q_{il} Q_{jl} Q_{kl} \neq 0 \end{cases}$$

- (2) *Given a decomposition $N = N_1 + \dots + N_k$, for the matrix $Q = \text{diag}(F_{N_1}, \dots, F_{N_k})$, where $F_N = \frac{1}{\sqrt{N}}(\xi^{ij})_{ij}$ with $\xi = e^{2\pi i/N}$ is the Fourier matrix, we obtain:*

$$\Gamma_Q = \mathbb{Z}_{N_1} * \dots * \mathbb{Z}_{N_k}$$

- (3) *Given an arbitrary matrix $Q \in U_N$, there exists a decomposition $N = N_1 + \dots + N_k$, such that Γ_Q appears as quotient of $\mathbb{Z}_{N_1} * \dots * \mathbb{Z}_{N_k}$.*

Proof. Here (1) was obtained in [1], via a computation that we will generalize later on, and (2) was obtained as well in [1], via a direct computation. As for (3), the result here goes back to Bichon’s work in [7], and can be obtained as well from (1,2). See [1]. \square

The above result is quite interesting for us, because it shows that the fibers Γ_Q not only wildly vary with Q , but are also not subject to quotient maps between them. This phenomenon holds of course as well for S_N itself, but the extension to S_N^+ is of interest, because at $N \geq 4$ this quantum group is known to be of “continuous” nature.

3. THE CONJECTURES

We present now our series of conjectures, relating the various algebraic and analytic properties of G to those of the family $\{\widehat{\Gamma}_Q | Q \in U_N\}$. As already mentioned in the introduction, while the general philosophy is quite old, going back to [1], [6], the conjectures are new, based on a quite substantial amount of recent work in the area.

Let us first recall that, according to the Peter-Weyl type theory in [29], the characters of finite dimensional representations of G live in a certain subalgebra $C(G)_{\text{central}} \subset C(G)$. To be more precise, $C(G)_{\text{central}}$ is by definition the norm closure of the linear span of the characters of irreducible representations, known to be linearly independent.

We recall as well that G is said to be connected if it has no finite quantum group quotient $G \rightarrow F \neq \{1\}$. This is equivalent to the fact that the coefficient algebra $\langle r_{ij} \rangle$ is infinite dimensional, for any finite dimensional unitary representation r . For group duals, $G = \widehat{\Gamma}$, this is the same as asking for Γ to have no torsion. See [11].

With this convention, our first conjecture is as follows:

Conjecture 3.1 (Characters). *If G is connected, for any nonzero $P \in C(G)_{\text{central}}$ there exists $Q \in U_N$ such that $\pi_Q : C(G) \rightarrow C^*(\Gamma_Q)$ has the property $\pi_Q(P) \neq 0$.*

Observe that this conjecture holds trivially when $G = \widehat{\Gamma}$ is a group dual, because we can take here $Q \in U_N$ to be the spinning matrix which produces the embedding $\widehat{\Gamma} \subset U_N^+$, coming from Theorem 1.5 above, and we have $\pi_Q = \text{id}$ in this case.

The conjecture holds as well in the classical group case, because we can take here $Q \in U_N$ to be such that $QTQ^* \subset \mathbb{T}^N$, where $T \subset U_N$ is a maximal torus for G .

Observe that in both the above cases, we have in fact a matrix $Q \in U_N$ such that π_Q is faithful on $C(G)_{\text{central}}$. In addition, the connectedness assumption is not really needed in the group dual case, nor for most of the known examples of compact groups. Thus, there are several potential ways of formulating some stronger conjectures.

At the analytic level now, our first, and main conjecture, will concern amenability. Let us recall from [29] that associated to any compact quantum group G are in fact several Hopf C^* -algebras, including a maximal one $C_{\text{max}}(G)$, and a minimal one $C_{\text{min}}(G)$. The compact quantum group G is said to be coamenable if the canonical quotient map $C_{\text{max}}(G) \rightarrow C_{\text{min}}(G)$ is an isomorphism. Equivalently, the discrete quantum group $\Gamma = \widehat{G}$ is called amenable when the canonical quotient map $C_{\text{max}}^*(G) \rightarrow C_{\text{min}}^*(G)$ is an isomorphism. With this convention, our conjecture is as follows:

Conjecture 3.2 (Amenability). *G is coamenable if and only if each Γ_Q is coamenable.*

Observe that \implies is trivial, because of the quotient map $C(G) \rightarrow C^*(\Gamma_Q)$, which can be interpreted as coming from a discrete quantum group quotient map $\widehat{G} \rightarrow \Gamma_Q$.

Regarding now \impliedby , an equivalent statement here, a bit more convenient, is that if G is not coamenable, then there exists $Q \in U_N$ such that Γ_Q is not amenable.

Observe that this latter statement holds trivially in the group dual case, $G = \widehat{\Gamma}$, because we can take here $Q \in U_N$ to be the spinning matrix coming from Theorem 1.5, for which $\pi_Q = id$. The statement holds as well in the classical case, $G \subset U_N$, due to the trivial fact that these latter quantum groups are all coamenable. See [29].

As already mentioned, the above conjecture is our main analytic one. We believe that the above statement is just the “tip of the iceberg”, with many other conjectures being behind it, some of them regarding the fine analytic structure of $C(G)$, and some other, regarding the fine probabilistic structure of the Kesten measure of \widehat{G} .

Regarding now the growth, let us recall from [4] that this is constructed by using the balls in $Irr(G)$, with respect to the distance coming from the fundamental corepresentation u , and with each corepresentation r contributing with a $\dim(r)^2$ factor.

With this convention, we have the following conjecture:

Conjecture 3.3 (Growth). *\widehat{G} has polynomial growth if and only if each Γ_Q has polynomial growth.*

As before, the conjecture is trivial in the group dual case. In the classical group case the conjecture holds as well, but this is not trivial, coming from [4] in the connected simply connected case, and from the recent paper [13] in the general case.

Once again, we believe that this conjecture is just the “tip of the iceberg”. Here is a series of more specialized statements, regarding the cardinality $|\cdot|$, the polynomial growth exponents $p(\cdot)$, and the exponential growth exponents $e(\cdot)$:

$$\begin{aligned} \log \log |G| &\simeq \sup_{Q \in U_N} \log \log |\Gamma_Q| \\ p(\widehat{G}) &\approx \sup_{Q \in U_N} p(\Gamma_Q) \\ \log e(\widehat{G}) &\approx \sup_{Q \in U_N} \log e(\Gamma_Q) \end{aligned}$$

These statements are all trivial in the group dual case, with equality everywhere. In the classical group case, the first estimate is what seems to come out from the existent literature, and the second estimate is non-trivial, but holds by [13]. Finally, regarding the last estimate, this is supported by the various computations in [4].

4. GENERAL RESULTS

We present here some general results, supporting the conjectures made in section 3. As a first statement, collecting the various observations made above, we have:

Proposition 4.1. *The 3 conjectures hold for the classical groups, and for the group duals.*

Proof. This follows as explained in the previous section, the summary being;

- (1) Characters: trivial for group duals, holds as well for classical groups.
- (2) Amenability: trivial for both group duals, and for classical groups.
- (3) Growth: trivial for group duals, nontrivial cf. [13] for classical groups. □

By getting back now to the precise justifications in section 3, observe that for the group duals, the argument was basically always the same, namely that we can take $Q \in U_N$ to be the spinning matrix coming from Theorem 1.5, for which $\pi_Q = id$.

Our aim now is to analyse and further extend this phenomenon. The definition that we will need, capturing the fact that “only one Q is needed”, is as follows:

Definition 4.2. *A compact quantum group $G \subset U_N^+$ is called “tame” when there exists $L \in U_N$ such that we have a quotient map $\Gamma_L \rightarrow \Gamma_Q$, for any $Q \in U_N$.*

Observe that, by changing the fundamental corepresentation, we can always assume that our tame quantum groups are “normalized”, with $L = 1$.

At the level of examples, any group dual is tame. Also, any compact connected Lie group is tame. As a basic counterexample, S_N is not tame, and nor is S_N^+ .

In the tame case, our conjectures have “lighter” formulations, as follows:

Proposition 4.3. *When G is tame and normalized, the conjectures are as follows:*

- (1) *Characters: $\pi_1 : C(G) \rightarrow C^*(\Gamma_1)$ is injective on $C(G)_{central}$.*
- (2) *Amenability: G is coamenable if and only if Γ_1 is amenable.*
- (3) *Growth: \widehat{G} has polynomial growth if and only if Γ_1 has polynomial growth.*

Proof. This follows indeed from definitions, by using the quotient maps $\Gamma_1 \rightarrow \Gamma_Q$ coming from the tameness assumption. □

It would be of course interesting to know more about the tame quantum groups. The problem here is that it is very unclear where the maps $\Gamma_L \rightarrow \Gamma_Q$ should come from.

Let us discuss now the verification for Wang’s free quantum groups O_N^+, U_N^+, S_N^+ . We include in our study the hyperoctahedral quantum group H_N^+ , which, in view of the general theory in [3], is a fundamental example of a free quantum group as well.

We have the following result, regarding these quantum groups:

Theorem 4.4. *The 3 conjectures hold for $G = O_N^+, U_N^+, S_N^+, H_N^+$.*

Proof. The idea is to use some special group dual subgroups. Observe first that for $G = O_N^+, U_N^+, S_N^+, H_N^+$ we have respectively $\Gamma_1 = \mathbb{Z}_2^{*N}, F_N, \{1\}, \mathbb{Z}_2^{*N}$. We will use these groups, along with some other groups of type Γ_Q , when needed.

We will need as well a number of standard isomorphisms, appearing at small values of N , namely $O_2^+ = SU_2^{-1}$, $S_2^+ = S_2$, $S_3^+ = S_3$, $S_4^+ = SO_3^{-1}$, $H_2^+ = O_2^{-1}$. These are in fact precisely the cases where G is coamenable. For details here, see [1], [9].

(1) Characters. For $G = O_N^+, U_N^+$, it is known that the algebra $C(G)_{central}$ is polynomial, respectively $*$ -polynomial, on the variable $\chi = \sum_i u_{ii}$. Thus, it is enough to show that the variable $\rho = \sum_i g_i$ generates a polynomial, respectively $*$ -polynomial algebra, inside the group algebra of the discrete groups \mathbb{Z}_2^{*N}, F_N . But for \mathbb{Z}_2^{*N} this is clear, and by using a multiplication by a unitary free from \mathbb{Z}_2^{*N} , the result holds as well for F_N .

Regarding $G = S_N^+$, at $N = 2, 3$ this quantum group is classical, equal to S_N , and the conjecture holds. At $N \geq 4$ the fusion rules are known to be the Clebsch-Gordan ones, and the algebra $C(G)_{central}$ is polynomial on $\chi = \sum_i u_{ii}$. In order to solve the problem at $N = 4$, observe that Theorem 2.6 gives, with $Q = \text{diag}(F_2, F_2)$, the discrete group $\Gamma_Q = \mathbb{Z}_2 * \mathbb{Z}_2 = D_\infty$. Since $\text{Tr}(u) = \text{Tr}(Q^*uQ)$, the image of $\chi = \sum_i u_{ii}$ in the quotient $C^*(\Gamma_Q)$ is the variable $\rho = 2 + g + h$, where g, h are the generators of the two copies of \mathbb{Z}_2 . Now since this latter variable generates a polynomial algebra, we obtain the result. Finally, at $N \geq 5$ the result follows by functoriality from the result at $N = 4$.

Regarding now $G = H_N^+$, here it is known from [5] that $C(G)_{central}$ is polynomial on the variables $\chi = \sum_i u_{ii}$ and $\chi' = \sum_i u_{ii}^2$. At $N = 2$ we have $H_2^+ = O_2^{-1}$, and, as explained in [1], with $Q = F_2$ we have $\Gamma_Q = D_\infty$. Let us compute now the images ρ, ρ' of the variables χ, χ' in the group algebra of D_∞ . As before, from $\text{Tr}(u) = \text{Tr}(Q^*uQ)$ we obtain $\rho = g + h$, where g, h are the generators of the two copies of \mathbb{Z}_2 . Regarding now ρ' , let us first recall that the quotient map $C(H_2^+) \rightarrow C^*(D_\infty)$ is constructed as follows:

$$\frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix}$$

Equivalently, this quotient map is constructed as follows:

$$\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \rightarrow \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} g+h & g-h \\ g-h & g+h \end{pmatrix}$$

We can now compute the image of our character, as follows:

$$\rho' = \frac{1}{2}(g+h)^2 = \frac{1}{2}(2 + 2gh) = 1 + gh$$

By using now the elementary fact that the variables $\rho = g + h$ and $\rho' = 1 + gh$ generate a polynomial algebra inside $C^*(D_\infty)$, this gives the result. Finally, at $N \geq 3$ the result follows by functoriality, via the inclusion $H_2^+ \subset H_N^+$, from the result at $N = 2$.

(2) Amenability. Here the cases where G is not coamenable are those of O_N^+ with $N \geq 3$, U_N^+ with $N \geq 2$, S_N^+ with $N \geq 5$, and H_N^+ with $N \geq 3$. For $G = O_N^+, H_N^+$ with $N \geq 3$ the result is clear, because $\Gamma_1 = \mathbb{Z}_2^{*N}$ is not amenable. Clear as well is the result for U_N^+ with $N \geq 2$, because $\Gamma_1 = F_N$ is not amenable. Finally, for S_N^+ with $N \geq 5$ the result holds as well, because of the presence of Bichon's group dual subgroup $\widehat{\mathbb{Z}_2 * \mathbb{Z}_3}$.

(3) Growth. Here the growth is polynomial precisely in the situations where G is infinite and coamenable, the precise cases being $O_2^+ = SU_2^{-1}$, $S_4^+ = SO_3^{-1}$, $H_2^+ = O_2^{-1}$, and the result follows from the fact that the growth invariants are stable by twisting.

Observe that for the remaining quantum groups, excluding the trivial examples $S_2^+ = S_2, S_3^+ = S_3$, the growth is exponential, and the estimate $\log e(\widehat{G}) \approx \sup_{Q \in U_N} \log e(\Gamma_Q)$ given at the end of section 3 is supported by the various computations in [4]. \square

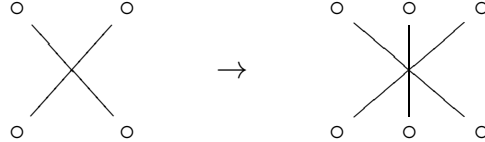
We can see from the above proof that the verification of the conjectures basically requires to know how to compute Γ_Q , and to know the representation theory of G .

There are many other situations where these two technical ingredients are available, at least to some extent. Without getting into details here, let us just mention that: (1) the product operations $\times, \hat{*}$ can be investigated by using [27], (2) the free complexification operation can be investigated by using [22], (3) for deformations, evidence comes from [9], [21], (4) for free wreath products, evidence comes from [18], (5) the two-parametric free quantum groups can be studied by using [2], and (6) for the various growth conjectures, substantial evidence comes from the computations in [4], [13].

In short, there is a lot of work to be done. In what follows we will do a part of this work, in relation with two key constructions, coming from Tannakian philosophy.

5. HALF-LIBERATION

One interesting discovery coming from Tannakian philosophy is the fact that the commutation relations $ab = ba$ can be successfully replaced, in relation with several quantum group questions, with the half-commutation relations $abc = cba$. Diagrammatically:



We use here the following notions, coming from [3], [6]:

Definition 5.1. *The half-classical orthogonal group O_N^* is given by:*

$$C(O_N^*) = C(O_N^+) / \left\langle abc = cba, \forall a, b, c \in \{u_{ij}\} \right\rangle$$

The closed quantum subgroups $G \subset O_N^$ are called half-classical.*

To be more precise now, this definition is motivated by a result from [6], stating that there are exactly 3 categories of pairings, namely those generated by \emptyset , \times , \bowtie . Equivalently, there are exactly 3 orthogonal easy quantum groups, namely:

$$O_N \subset O_N^* \subset O_N^+$$

We will be back to these topics in section 6 below. For the moment, we will just use Definition 5.1 as it is, and we refer to [3], [6] for where this definition comes from.

We will prove here that the 3 conjectures hold for any half-classical quantum group. In order to do so, we can use the modern approach from [8], which is as follows:

Proposition 5.2. *Given a conjugation-stable closed subgroup $H \subset U_N$, consider the algebra $C([H]) \subset M_2(C(H))$ generated by the following variables:*

$$u_{ij} = \begin{pmatrix} 0 & v_{ij} \\ \bar{v}_{ij} & 0 \end{pmatrix}$$

Then $[H]$ is a compact quantum group, we have $[H] \subset O_N^$, and any non-classical subgroup $G \subset O_N^*$ appears in this way, with $G = O_N^*$ itself appearing from $H = U_N$.*

Proof. The 2×2 matrices in the statement are self-adjoint, half-commute, and the $N \times N$ matrix $u = (u_{ij})$ that they form is orthogonal, so we have an embedding $[H] \subset O_N^*$. The quantum group property of $[H]$ is also elementary to check, by using an alternative, equivalent construction, with a quantum group embedding as follows:

$$C([H]) \subset C(H) \rtimes \mathbb{Z}_2$$

The surjectivity part is non-trivial, and we refer here to [8]. □

We will need as well the following result, also from [8]:

Proposition 5.3. *We have a bijection $\text{Irr}([H]) \simeq \text{Irr}_0(H) \coprod \text{Irr}_1(H)$, where*

$$\text{Irr}_k(H) = \left\{ r \in \text{Irr}(H) \mid \exists l \in \mathbb{N}, r \in u^{\otimes k} \otimes (u \otimes \bar{u})^{\otimes l} \right\}$$

induced by the canonical identification $\text{Irr}(H \rtimes \mathbb{Z}_2) \simeq \text{Irr}(H) \coprod \text{Irr}(H)$.

Proof. We have an equality of projective versions $P[H] = PH$, and so an inclusion $\text{Irr}_0(H) = \text{Irr}(PH) \subset \text{Irr}([H])$. The remaining irreducible representations of $[H]$ must come from an inclusion $\text{Irr}_1(H) \subset \text{Irr}([H])$, appearing as above. See [8]. □

Regarding now the maximal tori, the situation is very simple, as follows:

Proposition 5.4. *The group dual subgroups $\widehat{[\Gamma]}_Q \subset [H]$ appear via*

$$[\Gamma]_Q = [\Gamma_Q]$$

from the group dual subgroups $\widehat{\Gamma}_Q \subset H$ associated to $H \subset U_N$.

Proof. By using the crossed product picture, the operation $H \rightarrow [H]$ constructed in Proposition 5.2 is functorial, and in particular we have the following diagram:

$$\begin{array}{ccc} C([H]) & \subset & C(H) \rtimes \mathbb{Z}_2 \\ \downarrow & & \downarrow \\ C([\Gamma_Q]) & \subset & C(\Gamma_Q) \rtimes \mathbb{Z}_2 \end{array}$$

Thus we have an identification of discrete groups $[\Gamma]_Q = [\Gamma_Q]$, as claimed. □

Now back to our conjectures, we have the following result:

Theorem 5.5. *The 3 conjectures hold for any half-classical quantum group.*

Proof. We already know from Proposition 4.1 that the conjectures hold for any closed subgroup $G \subset O_N$, so we can assume that we have $G \subset O_N^*$, $G \not\subset O_N$.

According to Proposition 5.2, we have $G = [H]$, for a certain conjugation-stable subgroup $H \subset U_N$. By using once again Proposition 4.1, we know that the conjectures hold for $H \subset U_N$. The idea will be that of “transporting” these results, via $H \rightarrow [H]$:

(1) Characters. Assuming that $[H] \subset O_N^*$ is connected, it follows from Proposition 5.3 that $H \subset U_N$ is connected as well. Thus, we can pick a maximal torus $T = \Gamma_Q$ for the compact group $H \subset U_N$, and by using the formula $[\Gamma]_Q = [\Gamma_Q] = [T]$ from Proposition 5.4, we obtain the result, via the identification in Proposition 5.3.

(2) Amenability. There is nothing to be proved here, because O_N^* is coamenable, and so are all its quantum subgroups. Note however, in relation with the comments made in section 3 above, that in the connected case, the Kesten measures of $G, [T]$ are intimately related. For some explicit formulae here, for $G = O_N^*$ itself, see [3].

(3) Growth. Here the situation is similar, because by Proposition 5.3 above, $[H]$ has polynomial growth. As a comment, one can transport the results from [13], via the identification in Proposition 5.3, as to obtain precise exponent estimates. \square

Summarizing, we have extended to the half-classical setting the results regarding the compact groups $G \subset O_N$. Interesting would be to have a full extension of the classical results, with a statement covering all the closed subgroups $G \subset U_N$. This looks possible, but the general half-liberation theory here is not available yet.

6. TANNAKIAN ASPECTS

In this section we present a systematic Tannakian approach to our various conjectures. Our starting point is the following result, coming from Woronowicz’s work in [30]:

Proposition 6.1. *Given an inclusion $G \subset O_N^+$, with the corresponding fundamental corepresentations denoted $u \rightarrow w$, we have the following formula:*

$$C(G) = C(O_N^+) / \left(T \in \text{Hom}(u^{\otimes k}, u^{\otimes l}), \forall k, l \in \mathbb{N}, \forall T \in \text{Hom}(w^{\otimes k}, w^{\otimes l}) \right)$$

A similar result holds in the unitary case, by assuming that k, l are “colored” integers, with the tensor powers $v^{\otimes k}, v^{\otimes l}$ being obtained by tensoring v, \bar{v} .

Proof. This follows indeed from [30]. For a short, recent proof, see [19]. \square

Regarding now the tori, at this level of generality, we have the following result:

Proposition 6.2. *The intertwining formula $T \in \text{Hom}(u^{\otimes k}, u^{\otimes l})$, with $u = QvQ^*$, where $v = \text{diag}(g_1, \dots, g_N)$, is equivalent to the collection of conditions*

$$(T^Q)_{j_1 \dots j_l, i_1 \dots i_k} \neq 0 \implies g_{i_1} \dots g_{i_k} = g_{j_1} \dots g_{j_l}$$

one for each choice of the multi-indices i, j , where $T^Q = (Q^)^{\otimes l} T Q^{\otimes k}$.*

Proof. Observe first that, by conjugating by Q , we have the following formula:

$$T \in \text{Hom}(u^{\otimes k}, u^{\otimes l}) \iff T^Q \in \text{Hom}(v^{\otimes k}, v^{\otimes l})$$

Thus, it is enough to prove the result at $Q = 1$. And here, with standard multi-index notations, including the convention $g_i = g_{i_1} \dots g_{i_k}$, the computation goes as follows:

$$\begin{aligned} T \in \text{Hom}(u^{\otimes k}, u^{\otimes l}) &\iff Tu^{\otimes k}e_i = u^{\otimes l}Te_i, \forall i \\ &\iff Te_i \otimes g_i = u^{\otimes l} \sum_j T_{ji}e_j, \forall i \\ &\iff \sum_j T_{ji}e_j \otimes g_i = \sum_j T_{ji}e_j \otimes g_j, \forall i \\ &\iff T_{ji}g_i = T_{ji}g_j, \forall i, j \\ &\iff [T_{ji} \neq 0 \implies g_i = g_j], \forall i, j \end{aligned}$$

Thus we have obtained the relation in the statement, and we are done. \square

In principle Propositions 6.1 and 6.2 give all the needed ingredients for a Tannakian approach to our conjectures. Obviously, there is a lot of work to be done here.

Let us discuss now the easy case, where more concrete results can be obtained. We use the framework of [26]. Let $P(k, l)$ be the set of partitions between an upper row of k points, and a lower row of l points, with each leg colored black or white, and with k, l standing for the corresponding “colored integers”. We have then:

Definition 6.3. *A category of partitions is a collection of sets $D = \bigcup_{kl} D(k, l)$, with $D(k, l) \subset P(k, l)$, which contains the identity, and is stable under:*

- (1) *The horizontal concatenation operation \otimes .*
- (2) *The vertical concatenation \circ , after deleting closed strings in the middle.*
- (3) *The upside-down turning operation $*$ (with reversing of the colors).*

As explained in [19], [26], such categories produce quantum groups. To be more precise, associated to any partition $\pi \in P(k, l)$ is the following linear map:

$$T_\pi(e_{i_1} \otimes \dots \otimes e_{i_k}) = \sum_{j: \ker(\binom{i}{j}) \leq \pi} e_{j_1} \otimes \dots \otimes e_{j_l}$$

Here the kernel of a multi-index $\binom{i}{j} = \binom{i_1 \dots i_k}{j_1 \dots j_l}$ is the partition obtained by joining the sets of equal indices. Thus, the condition $\ker(\binom{i}{j}) \leq \pi$ simply tells us that the strings of π must join equal indices. With this construction in hand, we have:

Definition 6.4. *A compact quantum group $G \subset U_N^+$ is called easy when*

$$\text{Hom}(u^{\otimes k}, u^{\otimes l}) = \text{span} \left(T_\pi \Big| \pi \in D(k, l) \right)$$

for any k, l , for a certain category of partitions $D \subset P$.

In other words, the easiness condition states that the Schur-Weyl dual of G comes in the simplest possible way: from partitions. As a basic example, according to an old result of Brauer [10], the group $G = U_N$ is easy, with $D = P_2$ being the category of color-matching pairings. Easy as well is U_N^+ , with $D = NC_2 \subset P_2$ being the category of noncrossing color-matching pairings. See [3], [6], [15], [19], [24], [26].

With these conventions, we have the following result:

Theorem 6.5. *In the uncolored case, the intertwining formula $T_\pi \in \text{Hom}(u^{\otimes k}, u^{\otimes l})$, with $u = QvQ^*$, where $v = \text{diag}(g_1, \dots, g_N)$, is equivalent to*

$$\delta_\pi^Q \begin{pmatrix} i_1 \dots i_k \\ j_1 \dots j_l \end{pmatrix} \neq 0 \implies g_{i_1} \dots g_{i_k} = g_{j_1} \dots g_{j_l}$$

with the generalized Kronecker symbols being given by $\delta_\pi^Q = \prod_{\beta \in \pi} \delta_\beta^Q$, with:

$$\delta_{1_p^r}^Q \begin{pmatrix} i_1 \dots i_r \\ j_1 \dots j_p \end{pmatrix} = \sum_s Q_{si_1} \dots Q_{si_r} \bar{Q}_{sj_1} \dots \bar{Q}_{sj_p}$$

A similar result holds in the colored case, with the convention $g_\bullet = g^{-1}$.

Proof. With multi-index notations, as in the proof of Proposition 6.2, we have:

$$\begin{aligned} T_\pi^Q(e_i) &= (Q^*)^{\otimes l} T_\pi Q^{\otimes k} e_i \\ &= \sum_s (Q^*)^{\otimes l} T_\pi (Q^{\otimes k})_{si} e_s \\ &= \sum_s (Q^*)^{\otimes l} (Q^{\otimes k})_{si} \sum_t \delta_\pi \begin{pmatrix} s \\ t \end{pmatrix} e_t \\ &= \sum_{stj} \delta_\pi \begin{pmatrix} s \\ t \end{pmatrix} (Q^{\otimes k})_{si} ((Q^*)^{\otimes l})_{jt} e_j \end{aligned}$$

Thus, with full indices now, we have the following formula:

$$(T_\pi^Q)_{j_1 \dots j_l, i_1 \dots i_k} = \sum_{s_1 \dots s_k} \sum_{t_1 \dots t_l} \delta_\pi \begin{pmatrix} s_1 \dots s_k \\ t_1 \dots t_l \end{pmatrix} Q_{s_1 i_1} \dots Q_{s_k i_k} \bar{Q}_{t_1 j_1} \dots \bar{Q}_{t_l j_l}$$

Since this quantity is multiplicative with respect to the blocks of π , by decomposing over these blocks, we obtain the formula in the statement. \square

Observe that the above result generalizes the computation in Theorem 2.6. In view of the similarities with Proposition 2.3, one interesting question, that we would like to raise here, is that of extending Theorem 6.5, as to cover Proposition 2.3 as well.

At a more concrete level now, the orthogonal easy quantum groups were classified by Raum and Weber in [24]. Without getting into details here, let us mention that:

- (1) The classical examples are covered by Proposition 4.1. The free examples consist of the quantum groups S_N^+, H_N^+, O_N^+ from Theorem 4.4, and then of $S_N'^+, B_N^+$ and $B_N'^+, B_N''^+$, which appear as free versions of $S_N' = S_N \times \mathbb{Z}_2, B_N \simeq O_{N-1}$ and of $B_N' \simeq O_{N-1} \times \mathbb{Z}_2$, taken twice, where the methods for S_N^+, O_N^+ apply.
- (2) Then, we have a number of half-liberations, covered by Theorem 5.5, and an uncountable family, constructed in [23]. For this latter family we can use the diagonal group dual subgroup $\widehat{\Gamma}_1$, and by using the crossed product picture in [23] we conclude that our various conjectures hold indeed.
- (3) Finally, we have a last series, constructed in [24]. Here the quantum groups are not coamenable, and nor are their diagonal group dual subgroups $\widehat{\Gamma}_1$, so the amenability and growth conjectures are both satisfied. The remaining problem regards the character conjecture, and we have no results here.

In the easy unitary case, where the classification so far is only available in the classical and free cases [26], the situation is quite unclear. Regarding amenability, an idea here would be that of trying to relate the Kesten coamenability of G to the random walk on the various groups Γ_Q , with the restrictions on these latter random walks coming from the formula in Theorem 6.5. However, this looks like a quite technical task.

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